Developing adaptive expertise: A feasible and valuable goal for (elementary) mathematics education?[1]

Lieven Verschaffel

Koen Luwel

Joke Torbeyns

Wim Van Dooren

Center for Instructional Psychology and Technology
Katholieke Universiteit Leuven (Belgium)

Abstract: Some years ago, Hatano differentiated between routine and adaptive expertise and made a strong plea for the development and implementation of learning environments that aim at the latter type of expertise and not just the former. In this contribution we reflect on one aspect of adaptivity, namely the adaptive use of solution strategies in elementary school arithmetic. In the first part of this article we give some conceptual and methodological reflections on the adaptivity issue. More specifically, we critically review definitions and operationalisations of strategy adaptivity that only take into account task and subject characteristics and we argue for a concept and an approach that also involve the sociocultural context. The second part comprises some educational considerations with respect to the questions why, when, for whom, and how to strive for adaptive expertise in elementary mathematics education

***Key Word**: elementary learning; adaptative expertise; aritmetic; mathematical strategies. Enseñanza Primaria; expertise adaptativo; aritmètica; estrategias matemàticas.*

Resumen: Hace algunos años, Hatano diferenció entre el nivel de experto de aprendizaje de rutina y el adaptativo e impulsó el desarrollo y la implementación de ambientes de aprendizaje que tengan como objetivo el tipo adaptativo de experiencia por sobre el de rutina. En esta contribución, nos enfocamos en un aspecto de la adaptabilidad, la llamada adaptativa del uso de estrategias de solución en aritmética de educación primaria. En la primera parte de este artículo, brindamos algunos enfoques conceptuales y metodológicos en los asuntos de adaptabilidad. Más específicamente, hacemos una revisión crítica de las definiciones y operacionalizaciones de estrategias de adaptabilidad que solo toman en cuenta tareas y características del sujeto. Argumentamos la necesidad de un concepto y un enfoque que también incluya el contexto sociocultural. La segunda parte, comprende algunas consideraciones educativas con respecto a las preguntas de porque, cuando, para quien o cómo se procura la búsqueda de nivel de experto adaptativa en la educación de las matemáticas para la escuela primaria.

***Palabras Clave:** enseñanza primaria; expertise adaptativo; aritmètica; estrategias matemáticas.*

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Correspondence: Prof. Dr. Lieven Verschaffel, Centre for Instructional Psychology and Technology, Vesaliusstraat 2, 3000 Leuven, Belgium. Tel: +32 16/326258, fax: +32 16/32.62.74, e-mail: lieven.verschaffel@ped.kuleuven.be.

Introduction

In the beginning of 2006 the international research community of developmental and educational psychologists lost one of its greatest scholars, Giyoo Hatano. In his foreword to the book *The development of arithmetic concepts and skills. Constructing adaptive expertise* edited by Baroody and Dowker (2003), the late Hatano (2003, p. xi) argues that one of the most important issues is how students can be taught curricular subjects so that they develop *adaptive expertise*. He describes *adaptive expertise* as "the ability to apply meaningfully learned procedures flexibly and creatively" and opposes it to *routine expertise*, i.e. "simply being able to complete school mathematics exercises quickly and accurately without (much) understanding".

Although the constructs of adaptive and routine expertise were introduced by Hatano already more than two decades ago (Hatano, 1982) and although terms like adaptivity and flexibility – which we will treat as synonyms in this article - have been used with increasing frequency by researchers and practitioners in the field of mathematics education for a long time, few attempts have been made to rigorously and systematically study (a) adaptive expertise as a competence, (b) the acquisition of adaptive expertise, and (c) its cultivation, and (d) its assessment, in the academic domain of (elementary) mathematics.

Of course, the older and recent histories of the research fields of (cognitive) psychology and (mathematics) education comprise numerous valuable building blocks for such an endeavor. For instance, there is the classical work of Gestalt psychologists like Duncker (1945) on 'functional fixedness' and Luchins and Luchins (1959) on 'problem-solving sets', which provides strong evidence for the rigidifying effects of long-term stereotyped practice, as well as Wertheimer's (1945) richly documented contrastive analysis between blind following of procedures and detection and exploitation of (mathematical) structure. Flexibility played also a central role in Krutetskii's (1976) famous study of mathematical ability and in several other analyses of individual differences in ability and creativity in general (see e.g., Guilford, 1967). Furthermore, we refer to studies of the thinking processes of eminent mathematicians (see e.g. Cajori, 1917; Wertheimer, 1945) or skilled calculating prodigies (see Heavey, 2003) demonstrating remarkable flexibility in their thinking. Finally, mathematics educators like Freudenthal (1983) and Treffers, De Moor and Feijs (1990) in the Netherlands and Wittmann and Müller (1990-1992) in Germany have already for a long time emphasized the educational importance of recognizing and stimulating the variety and flexibility in children's self-constructed strategies as a major pillar of their innovative approaches of (elementary) mathematics education.

The rise of cognitive psychology during the last decades of the previous century has contributed to a more fine-grained understanding of the cognitive processes and mechanisms underlying flexibility. Siegler's series of computer simulation models of how children choose adaptively among strategies for particular tasks in elementary arithmetic probably constitutes the most ambitious and explicit theoretical account (from the cognitive-psychological perspective) of procedural flexibility in (elementary) arithmetic (Shrager & Siegler, 1998).

While there is little doubt that cognitive psychology has significantly improved our understanding of flexibility and its role in mathematical expertise, recent developments within our research field have revealed that the quintessence of flexibility cannot be grasped in (purely) cognitive terms alone. In their discussion paper at the end of a special issue of the *Educational Researcher* on 'Expertise', Hatano and Oura (2003) emphasize that the process of gaining adaptive expertise always occurs in particular sociocultural contexts and is accompanied by changes in interest, values and identity (besides changes in the cognitive realm). This emphasis is echoed in Boaler's (2000, p. 118) plea for a greater consideration of the 'macro context' in which (math) educational research is conducted, and for "extending our focus beyond the concepts and procedures that pupils learn to the practices in which they engage as they are learning them and the mediation of cognitive forms by the environments in which they are produced".

In the first part of this article we will give some further conceptual and methodological reflections on the adaptivity issue, while in the second part of it, we will give some educational considerations. But before starting, we would like to emphasize that the opposition between routine and adaptive expertise does, of course, not apply only to the use of mathematical procedures or strategies -

which will be the focus of this article -, but to other aspects of mathematical expertise. For instance, it can also be applied to mathematical models (= modeling flexibility) and representational forms (= representational flexibility).

Some conceptual and methodological considerations

Sometimes procedural flexibility or adaptivity is simply identified with using a variety of solution strategies. For instance, at the beginning of their ascertaining study of the role of flexibility in accurate mental addition and subtraction in the number domain up to 100, Heirdsfield and Cooper (2002, p. 59) raise the following question: "Why some students are more accurate and flexible (= using a variety of strategies) than others?" Further on, they distinguish between pupils who are 'accurate and flexible' and others who are 'accurate and inflexible' merely on the basis of the question whether they did or did not solve all items from a given problem set by means of the same procedure. Although possessing a variety of strategies can be considered a 'conditio sine qua non' for procedural flexibility or adaptivity, merely showing that a group of children or even an individual child applies a greater variety of solution strategies on a series of tasks than another group of children or another individual child can hardly be considered as a (strong) proof of it. So, while variety in one's strategy repertoire is an important parameter of strategic competence, it is a different one than, and only a precondition for, adaptivity. Indeed, most people will agree that one can use a variety of procedures without acting adaptively, but also that using consistently one single strategy for a whole series of arithmetic tasks might sometimes be more adaptive than switching between a diversity of strategies available in one's repertoire.

More frequently, procedural flexibility has been defined in relation to certain task characteristics. For instance, Van der Heijden (1993, p. 80) defines it as follows: "Flexibility in strategy use involves the flexible adaptation of one's solution procedures to task characteristics". He operationalized flexibility by analyzing whether children systematically use the 1010-procedure and the G10-procedure for, respectively, additions and subtractions in the number domain up to 100[2] . Starting from exactly the same definition, Blöte, Van den Burg and Klein (2001, p. 628) operationalize flexibility of strategy use in addition and subtraction with numbers up to 100 as follows: "A student is considered a flexible problem solver if he or she chooses the solution procedures in relation to the number characteristics of the problem, for example, N10C for 62-29 and A10 for 62-24[3] ". In other words, these authors first distinguish different procedures for doing additions and subtractions up to 100, and, based on an analysis of the strengths and weaknesses of these different strategies vis-à-vis certain types of problems, they define certain 'problem type x strategy type combinations' as flexible and others as inflexible. Although this operationalization of adaptivity can already be considered as more sophisticated than the one wherein adaptivity is simply identified with strategy variety, it remains highly problematic to evaluate procedural flexibility in terms of task characteristics alone. Indeed, it is possible that, for a particular subject and/or under particular circumstances, the strategy choice process that Van der Heijden (1993) and Blöte et al. (2001) call 'flexible' may become 'inflexible', and vice versa. Hereafter, we consider two other kinds of factors that need to be incorporated in a comprehensive concept of adaptivity, besides task variables, namely subject and context variables.

A first set of complicating factors, which has been intensively and systematically investigated and modeled by cognitive psychologists like Siegler (1998, see also Shrager & Siegler, 1998), are the subject variables. According to SCADS (Strategy Choice and Discovery Simulation), Siegler's latest computer model of how children's mastery of simple arithmetic sums develops, whether a particular strategy (e.g. retrieval or a particular counting strategy) is chosen to solve a particular item by a particular child depends on how accurate and how quick that strategy is

[2] The 1010 procedure involves splitting off the tens and the units in both integers and handling them separately (e.g., 47+15=.; 40+10=50; 7+5=12; 50+12=62). The application of the G10 procedure, on the other hand, requires the child to add or subtract the tens and the units of the second integer to resp. from the first unsplit integer (e.g., 47+15=.; 47+10=57; 57+5=62).

[3] Using N10C for 62-29 means that the student does "62-30= 32, then 32+1=33, while a A10 strategy for 62-24 involves the following steps: "62-2=60, 60-20=40, 40-2=38".

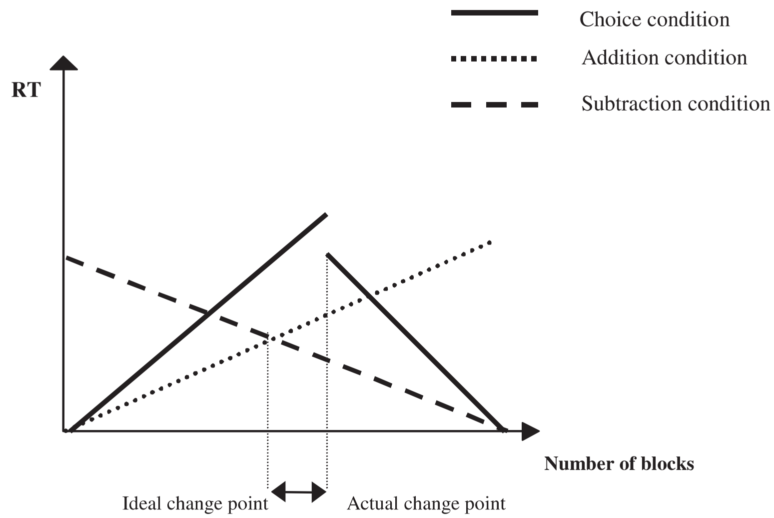
for that particular item and for that particular child, in comparison to other concurrent strategies available in that child's repertoire. So, Siegler's computer simulation model always tends to select and apply the strategy that produces the most beneficial combination of speed and accuracy for a particular sum. Undoubtedly, the adaptivity concept underlying this computer model reflects a more complex and more subtle view on the strategy choice process, wherein affordances inherent in the task have to be seen in relation to, and balanced with, subject characteristics of the individual who is solving the task.

Of course, conceptualizations of strategy choice that comprise both task and subject variables ask for capturing this assumed complexity of the strategy choice process in the research methodology. A method that is being increasingly used for this purpose is the choice/no-choice method (Siegler & Lemaire, 1997). The choice/no-choice method requires testing each subject under two types of conditions. In the choice condition, subjects can freely choose which strategy they use to solve each problem. In the no-choice condition, subjects must use one particular strategy to solve all problems. The number of no-choice conditions equals the number of strategies available in the choice condition. The obligatory use of one particular strategy on all problems in the no-choice condition by each participant allows the researcher to obtain unbiased estimates of the speed and accuracy of the strategy. Comparison of the data about the accuracy and the speed of the different strategies as gathered in the no-choice conditions, with the strategy choices made in the choice condition, allows the researcher to assess the adaptiveness of individual strategy choices in the choice condition in a scientifically appropriate way: Does the subject (in the choice condition) solve each problem by means of the strategy that yields the best performance – in terms of accuracy and speed - on this problem, as evidenced by the information obtained in the no-choice conditions? Over the past few years our team has also realized several studies wherein this method has been successfully used to investigate the adaptivity of elementary children's strategy choices in domains like elementary addition and subtraction and numerosity estimation (Torbeyns, Verschaffel & Ghesquière, 2004; Luwel, Verschaffel & Lemaire, 2005). For instance, in one our research programs, we investigate people's strategy choices for judging numerosities of coloured blocks presented in rectangular grids (e.g. in a 10 x 10 grid). Basically, we claim that there are two main strategies to solve this task: an *addition strategy* (wherein the total number of blocks in the grid is divided into subgroups and the number of blocks in each subgroup is added to a running total), which seems effective for small numerosities, and a *subtraction strategy* (in which the (estimated) number of empty squares is subtracted from the total number of squares in the grid, e.g., in the above example, from 100), which is effective for larger numerosities. Moreover, we assume that each strategy has a specific course of response times as a function of the numerosity of blocks in the grid (from 1 until 100, in the case of a 10 x 10 grid). Using the choice/no-choice method, we were able to show that as students get older and more experienced, they become more adaptive in their choice for an addition or a subtraction strategy, in the sense that, with growing age, we observed a decrease in the distance between a) the numerosity at which they stop using addition and start using subtraction in the choice condition and b) the numerosity at which, according to the reaction time and accuracy data in the two no-choice conditions, it would be most efficient to exchange the addition for the subtraction strategy (see Figure 1). So, this distance between these two change points in the reaction time data patterns, which we considered as a nice measure of individuals' strategy adaptiveness, became considerably smaller as the students grew older (Luwel et al., 2005).

As already stated in the introductory section of this article, recent theoretical developments, and especially the rise of socio-cultural views, have revealed that the issue of adaptivity is even more complicated than suggested by cognitive computer models like the one by Siegler c.s. Applied to his SCADS model, the question is whether there is not a set of situational or contextual variables that also bear influence on the strategy choice process (but that do not lend themselves easily to experimental manipulation or control), like students' needs to make sense of the task, their attempts to meet the (implicit) expectations of the teacher or the researcher, and the broader sociocultural or instructional context wherein the child has to select and execute an arithmetic strategy (Ellis, 1997).

A first nice example is the well-known study by Nunes, Carraher and Schliemann (1993),

Figure 1. Schematic presentation of the adaptivity measure used by Luwel et al. (2005), i.e. the distance between the actual change point in the choice condition and the ideal change point inferred from the integration of the data from the addition and subtraction no-choice conditions, as indicated by the length of the arrow below the horizontal axis (from: Luwel et al., 2005).



showing that whether the problem is presented as a real-world problem in an out-of-school context or as an arithmetic task in a school context (co-)determines the kind of arithmetic strategy - oral versus written arithmetic – children select and use to solve the problem.

As a second example, we refer to a study by Carr and Jessup's (1997), who found gender-related differences in the arithmetic strategies used by first-grade children (e.g. they found that boys started earlier to exchange counting for retrieval strategies than girls) that were significantly associated with differences in the nature and the importance of the girls' and boys' beliefs about what type of strategies would be valued most (as indications of ability) by their teacher and their parents.

So, people's strategy choices in elementary arithmetic are not only determined by task and subject variables, but also by characteristics of the environment wherein they have to demonstrate their arithmetic skills, including what representational and computational tools are available and allowed and/or what aspects of their strategic behavior – speed, correctness, certitude, simplicity, efficiency, elegance, formality or generality of the solution strategy – seem (most) valued in the classroom and/or testing context (Ellis, 1997).

The methodological implication of this theoretical complication is that research should also try to take these complicating contextual factors also into account when investigating (the development of) procedural flexibility. Possible methodological steps into that direction are:

- use of individual interviews to get deeper insight into subjects' perception and interpretation of the context wherein they have to make strategy choices and how this influences their strategy choices;
- systematic manipulation of the 'experimental contract' (e.g., by explicitly or implicitly rewarding particular aspects of one's strategic behavior, like the correctness, speed, cleverness, originality, etc. of the solution process);
- application of video-based observations of how children (learn to) make strategy choices in real classroom settings, – if possible, complemented with (video-based stimulated) interviews afterwards.

This recommendation to pay more attention to contextual c.q. instructional variables is echoed in Bisanz' (2003, p. 447) commentary at the end of Baroody and Dowker's (2003a) book, wherein he makes a strong plea for integrating the analysis of instructional materials and of teacher behavior into the study of acquisition of (adaptive) expertise in elementary arithmetic. Referring to one of the chapters in the book, Bisanz writes: "Seo and Ginsburg demonstrate the richness of insight that can arise when researchers analyze not only the responses or interpretations of children but also the environments in which those children develop. In this case the analysis of textbooks and teacher behavior were essential to gaining insights that otherwise would have been regarded only as speculative. We would like to see more of this sort of sensitivity and thoroughness in the mainstream of developmental research". While fully endorsing Bisanz' plea, we would like to take it even a step further and add that research on procedural flexibility should not only investigate these instructional environments in which this flexibility develops by means of 'ecologically valid' ascertaining studies, but also, and may-be even primarily, through design experiments, wherein researchers work at "the construction of 'artificial objects', namely teaching units, sets of coherent teaching units and curricula as well as the investigation of their possible effects in different educational 'ecologies'", as has been argued so eloquently and convincingly argued by Wittmann (1995, p. 363) in his seminal article *Mathematics education as a design science*.

Before closing the first part of this article, wherein we have worked towards the following definition of what it means to be adaptive in one's strategy choices: "*By adaptive or flexible use of mathematical strategies we mean the use of the most appropriate solution strategy on a given mathematical task, for a given individual, in a given context or situation",* we would like to give two short comments on this definition. Fist, as emphasized earlier, by 'the most appropriate strategy' we certainly do not simply mean 'the quickest strategy that leads to the correct answer' (as in the narrow cognitive-psychological sense of the word). Second, the above definition of flexibility does not involve any reference to the level of deliberateness and consciousness of the strategy choice being made. Many researchers believe that conscious awareness and deliberate control regulate strategy selection and, thus, procedural flexibility. But there is also evidence suggesting that, especially in quick and simple strategy selections like the ones addressed by SCADS, children's selection of a particular strategy does not result from deliberate consideration of the choices and from conscious awareness of the factors that influenced that choice, but rather from more autonomous, implicit processes. Evidently, the claim that the strategy selection process itself can be(come) automatized to such an extent that it escapes from the solver's conscious awareness and deliberate control, has important methodological – but also: educational - implications. This brings us to the second part of our presentation, wherein we will raise some educational considerations about the issue of procedural flexibility.

Educational considerations

Underlying many current curriculum reform documents, such as the *Curriculum and evaluation standards for school mathematics* of the National Council of Teachers of Mathematics in the US (1989, 2000), the *Numeracy Strategy* in the UK (Straker, 1999), the *Proeve van een Nationaal Programma voor het Reken/wiskundeonderwijs* in The Netherlands (Treffers et al., 1990), or the *Handbuch Produktiver Rechenübungen* (Wittmann & Müller, 1990) in Germany, and many innovative curricula, textbooks and other instructional materials more or less explicitly based on these reform documents, there is a basic belief in the feasibility and value of the strive for procedural flexibility. However, this basic belief, as well as some accompanying presuppositions about *what* kind of adaptive expertise we should strive for, and *when* and *for whom* and *how* to strive for it, have not yet been subjected to much systematic and scrutinized theoretical reflection and/or empirical research.

First, there is the issue of the optimal moment for beginning to strive for adaptive expertise. Several authors argue that one better teaches, first and above all, for routine expertise, and only afterwards, changes one's aims and pedagogy in the direction of adaptive expertise (see e.g. Geary, 2003; Milo & Ruijssenaars, 2002; Warner, Davis, Alcock, & Coppolo, 2002). This viewpoint is opposite to the view of most advocates of reform-based approaches to mathematics education, who conjecture that the development of adaptive expertise is not something that simply happens *after* people develop routine expertise and that education for adaptive expertise should be already present from the very beginning of the teaching/learning process (see e.g., Baroody, 2003; Gravemeijer, 1994; Selter, 1998; Wittmann & Müller, 1992), – – an idea that is nicely expressed in the following quote from Bransford (2001, p. 3): "You don't develop it in a 'capstone course' at the end of students' senior year. Instead the path toward adaptive expertise is probably different from the path toward routine expertise. Adapative expertise involves habits of mind, attitudes, and ways of thinking and organizing one's knowledge that are different from routine expertise and that take time to develop. We don't mean to imply that 'you can't teach an old routine expert new tricks'. But it's probably harder to do this than to start people down an 'adaptive expertise' path to begin with–at least for most people." There is a great need of comparative research involving distinct instructional approaches that differ with respect to the moment at which strategic adaptivity is aimed at and that involve comparisons of learning, retention, and transfer effects on a broad scale of cognitive as well as non-cognitive variables.

Closely related to this first issue is the question whether promoting variable and flexible strategy use is feasible and valuable across all levels of mathematical achievement, including the weaker ones. Many advocates of the reform movement believe this is the case (Baroody, 2003; Moser Opitz, 2001; Van den Heuvel-Panhuizen, 2001). Whereas some studies (e.g., Klein, Beishuizen & Treffers, 1998) indicate that mathematically weak children can also profit from instruction aiming at the flexible application of variable strategies, other investigations (e.g., Milo & Ruijssenaars, 2002) question the efficacy of this type of instruction for these children and seem to suggest that instruction aimed at procedural variety and flexibility will lead to a degradation in strategy efficiency, and, hence, in overall performance. While there are some indications that warrant optimism (see e.g., Baroody, 2003; Moser Opitz, 2001; Van den Heuvel-Panhuizen, 2001), more focused research is needed to determine whether it is indeed possible to design and implement instructional approaches aiming at (procedural) flexibility that are successful for all children, including the mathematically weaker ones.

This brings us to the third and last question, namely how to design and implement effective instruction aimed at adaptive expertise in (elementary) mathematics education. Assuming that children will profit from an approach wherein they are, from an early stage in their learning process on, confronted with a variety of methods rather than being provided with and trained in a single, uniform solution method, some reform-based curriculum developers and textbook authors provide children, in a non-evaluative way, an overview of several alternative solution methods for doing mental arithmetic (e.g., in the number domain 1-20, 20-100, or 100-1000) and stimulate them to select their own preferential strategy or strategies, and to talk about and reflect upon their selections. These curriculum developers or textbook authors do not try to establish fixed links between particular mental calculation strategies and particular problem types, nor try to develop in an explicit and systematic way some kind of overall strategy for selecting the most appropriate solution strategy for every problem type. Other curriculum developers and textbook authors take quite a different strategy and provide explicit and systematic teaching of different strategies for doing mental arithmetic in the above-mentioned number domains and teach children, afterwards, explicitly and systematically how to use each strategy 'adaptively'. More particularly, during several lessons children not only learn to identify, name and apply the different mental calculation strategies; they also learn to link each strategy to a particular type of sums which that strategy is considered (by the curriculum developers or the textbook authors) most efficient. Contrary to the first approach, *all* children are now provided by with the same method for identifying particular types of arithmetic problems and for applying *the* most efficient solution strategy for each problem type.

Based on the theoretical considerations provided in the first part of this article, it seems clear that providing children with a '(quasi-)algorithmic' rule for linking problem types to solution strategies and with systematic training in the use of that rule, as in the second example, is not the kind of instruction that will yield adaptive expertise as we have conceived and defined it. Such instruction, based on a notion of adaptive expertise that merely involves adaptivity to task variables without any consideration of individual or contextual factors, misjudges the quintessence of adaptive expertise, which must involve a personal and insightful choice based on weighing different kinds of affordances, – not only task-, but also subject- and context-related.

The more one dismisses a notion of procedural flexibility that merely looks at task characteristics, the more one will agree that there is no easy shortcut to learning to become adaptive, and that adaptive expertise is not something that can be *trained or taught*, but something that has to be *promoted* or *cultivated*. The use of these latter terms emphasizes, first, that the acquisition of adaptive expertise takes place in the sociocultural context of the classroom, second, that is as well a motivational and emotional matter as a cognitive one.

Hatano and Oura (2003) proposed the following inspiring set of motivational and affective conditions for placing students on a trajectory towards adaptive expertise (conceived in line with its broad definition given at the end of Section 2): (a) encountering novel problems continuously, (b) engaging in interactive dialogue, (c) being freed from urgent external need to perform, (d) being surrounded by a group that values understanding. We believe that these conditions - which were developed for other domains than mathematics and for learners of ages well beyond the elementary school level - are also critical for cultivating adaptive expertise in (elementary) mathematics education. But, this is only a belief, which needs to be subjected to rigorous empirical research.

So, if we want to make progress in our theoretical understanding and practical enhancement of procedural flexibility in elementary arithmetic, there is a great need of empirical studies – and especially of design experiments – that possess the following characteristics. First, these studies should be done in 'ecologically valid' settings, which means that they are done in regular classes with regular teachers, pupils and involve analyses of textbooks and teacher behavior over longer periods of time. Second, serious attention should be paid at the optimal moment to start with this strive for procedural flexibility and at the potentially differential impact on children of different (mathematical) abilities. Third, a good balance between the social and the individual perspective – or stated differently: between the socio-mathematical norms about what it means to be flexible, on the one hand, and the development of individual children's strategic flexibility, on the other hand – is required. Fourth, given that procedural flexibility in elementary arithmetic is – from a mathematics education perspective – not a goal in itself but mainly a vehicle for reaching more general and far-reaching goals - the assessment of children's development in these studies should not be restricted to procedural flexibility and competence 'an sich', but should also comprise their understanding of 'big ideas' and principles underlying the different strategies as well as their emotions, beliefs and attitudes towards flexibility and mathematics in general.

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